1. (a) Find $\int x^2 e^x \, dx$.

(5)

(b) Hence find the exact value of $\int_0^1 x^2 e^x \, dx$.

(2)

$$\int uv' = uv - \int u'v \qquad u = x^2 \quad v = e^x$$
$$u' = 2x \quad v' = e^x$$

$$\int x^2 e^x = x^2 e^x - 2\int x e^x$$

$$\int x e^x = x e^x - \int e^x \qquad u = x \quad v = e^x$$
$$u' = 1 \quad v' = e^x$$
$$= x e^x - e^x$$

$$\therefore \int x^2 e^x = x^2 e^x - 2(x e^x - e^x)$$
$$= (x^2 - 2x + 2)e^x + c \#$$

b) $\int_0^1 x^2 e^x = \left[(x^2 - 2x + 2)e^x \right]_0^1 = e^1 - 2$

$$= \underline{e - 2}$$

2. (a) Use the binomial expansion to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1 + x + \frac{1}{2}x^2, \quad |x| < 1$$

(6)

(b) Substitute $x = \frac{1}{26}$ into

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = 1 + x + \frac{1}{2}x^2$$

to obtain an approximation to $\sqrt{3}$

Give your answer in the form $\frac{a}{b}$ where a and b are integers.

(3)

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}} \times (1-x)^{-\frac{1}{2}}$$

$$(1+x)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2 \ldots$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(-x)^2$$

$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2$$

$\therefore$

$\times$	1	$+\frac{1}{2}x$	$-\frac{1}{8}x^2$
1	1	$\frac{1}{2}x$	$-\frac{1}{8}x^2$
$+\frac{1}{2}x$	$\frac{1}{2}x$	$\frac{1}{4}x^2$	$\times$
$+\frac{3}{8}x^2$	$\frac{3}{8}x^2$	$\times$	$\times$

$$= 1 + x + \tfrac{4}{8}x^2$$

$$= 1 + x^2 + \tfrac{1}{2}x^2 \quad \#$$

b) $$\sqrt{\frac{1+\frac{1}{26}}{1-\frac{1}{26}}} = \sqrt{\frac{\frac{27}{26}}{\frac{25}{26}}} = \sqrt{\frac{27}{25}} = \frac{3\sqrt{3}}{5} = \frac{3}{5}\sqrt{3}$$

$$\therefore \tfrac{3}{5} \times \sqrt{3} \simeq 1 + \tfrac{1}{26} + \tfrac{1}{2}\left(\tfrac{1}{26}\right)^2 = \frac{1405}{1352} \qquad \therefore \sqrt{3} = \frac{7025}{4056}$$

blank

3.

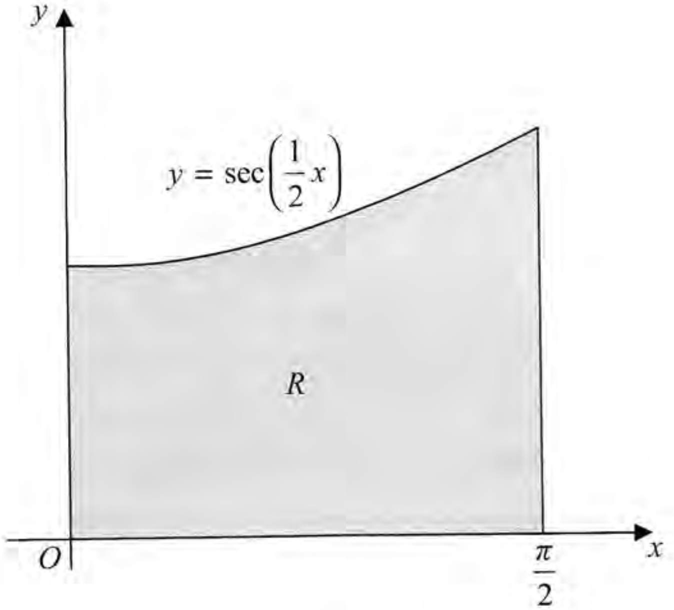


Figure 1

Figure 1 shows the finite region R bounded by the x-axis, the y-axis, the line $x = \frac{\pi}{2}$ and the curve with equation

$$y = \sec\left(\frac{1}{2}x\right), \quad 0 \leqslant x \leqslant \frac{\pi}{2}$$

The table shows corresponding values of x and y for $y = \sec\left(\frac{1}{2}x\right)$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	1	1.035276	1.154701	1.414214

(a) Complete the table above giving the missing value of y to 6 decimal places.
(1)

(b) Using the trapezium rule, with all of the values of y from the completed table, find an approximation for the area of R, giving your answer to 4 decimal places.
(3)

Region R is rotated through 2π radians about the x-axis.

(c) Use calculus to find the exact volume of the solid formed.
(4)

b) $\frac{1}{2}\left(\frac{\pi}{6}\right)\left[1+1.414214+2(1.035276+1.154701)\right] \simeq 1.7787$

c) $\text{Vol} = \pi\int_0^{\frac{\pi}{2}} \sec^2\left(\tfrac{1}{2}x\right)dx = \pi\left[\frac{1}{\frac{1}{2}}\tan\left(\tfrac{1}{2}x\right)\right]_0^{\frac{\pi}{2}} = \pi\left[2\tan\left(\tfrac{1}{2}x\right)\right]_0^{\frac{\pi}{2}} = \pi\left[2\tan\left(\tfrac{\pi}{4}\right)-0\right]$

$= \underline{2\pi}$

4. A curve C has parametric equations

$$x = 2\sin t, \quad y = 1 - \cos 2t, \quad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$

(a) Find $\dfrac{dy}{dx}$ at the point where $t = \dfrac{\pi}{6}$

(4)

(b) Find a cartesian equation for C in the form

$$y = f(x), \quad -k \leqslant x \leqslant k,$$

stating the value of the constant k.

(3)

(c) Write down the range of $f(x)$.

(2)

a) $x = 2\sin t \qquad y = 1 - \cos 2t$

$\dfrac{dx}{dt} = 2\cos t \qquad \dfrac{dy}{dt} = 2\sin 2t$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{2\sin 2t}{2\cos t} = \dfrac{4\sin t\cancel{\cos t}}{2\cancel{\cos t}}$

$= 2\sin t$

$\left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{6}} = 1$

b) $y = 1 - \cos 2t = 1 - (1 - 2\sin^2 t) = 1 - \left(1 - 2\left(\frac{x}{2}\right)^2\right)$

$= 2\dfrac{x^2}{4} = \dfrac{x^2}{2}$

$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \quad \therefore -1 \leq \sin t \leq 1 \Rightarrow -2 \leq x \leq 2 \quad \underline{k=2}$

c) $0 \leq y \leq 2$

5. (a) Use the substitution $x = u^2$, $u > 0$, to show that

$$\int \frac{1}{x(2\sqrt{x}-1)}\,dx = \int \frac{2}{u(2u-1)}\,du$$

(3)

(b) Hence show that

$$\int_1^9 \frac{1}{x(2\sqrt{x}-1)}\,dx = 2\ln\left(\frac{a}{b}\right)$$

where a and b are integers to be determined.

(7)

$x = u^2 \Rightarrow \sqrt{x} = u.$ $\Rightarrow \int \frac{1}{u^{\cancel{2}}(2u-1)} \cancel{2}u\,du$

$\frac{dx}{du} = 2u \Rightarrow dx = 2u\,du$

$= \int \frac{1}{u^{\cancel{2}}(2u-1)} 2\cancel{u}\,du = \int \frac{2}{u(2u-1)}\,du$ #

$x = 9 \quad u = 3$

$x = 1 \quad u = 1$

$\int_1^3 \frac{2}{u(2u-1)}\,du.$

$$\frac{2}{u(2u-1)} = \frac{A}{u} + \frac{B}{2u-1} = \frac{A(2u-1)+B(u)}{u(2u-1)}$$

$$\therefore\ 2 \equiv A(2u-1)+B(u) \qquad u=0 \Rightarrow \underline{A=-2}$$

$$u=\tfrac{1}{2} \Rightarrow 2=\tfrac{1}{2}B \quad \therefore \underline{B=4}$$

$$\int_1^3 \frac{-2}{u} + \frac{4}{2u-1}\,du = -2\int_1^3 \frac{1}{u}\,du + 2\int_1^3 \frac{2}{2u-1}\,du$$

$$= \Big[-2\ln u + 2\ln(2u-1)\Big]_1^3$$

$$= (-2\ln 3 + 2\ln 5) - (\cancel{-2\ln 1} + \cancel{2\ln 1})$$

$$= 2(\ln 5 - \ln 3) = 2\ln\left(\frac{5}{3}\right)$$

6. Water is being heated in a kettle. At time t seconds, the temperature of the water is θ °C.

The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{d\theta}{dt} = \lambda(120 - \theta), \qquad \theta \leqslant 100$$

where λ is a positive constant.

Given that $\theta = 20$ when $t = 0$,

(a) solve this differential equation to show that

$$\theta = 120 - 100e^{-\lambda t}$$

(8)

When the temperature of the water reaches 100 °C, the kettle switches off.

(b) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off.

(3)

a) $-\int \frac{-1}{120-\theta}\, d\theta = \int \lambda\, dt$

$\Rightarrow -\ln(120-\theta) = \lambda t + c$

$\theta = 20, t = 0 \quad \Rightarrow \quad -\ln 100 = c$

$\Rightarrow -\ln(120-\theta) = \lambda t - \ln 100$

$\Rightarrow \ln(100) - \ln(120-\theta) = \lambda t \quad \Rightarrow \ln\left(\frac{100}{120-\theta}\right) = \lambda t$

$\Rightarrow \frac{100}{120-\theta} = e^{\lambda t} \quad \Rightarrow \frac{100}{e^{\lambda t}} = 120 - \theta$

$\Rightarrow \theta = 120 - \frac{100}{e^{\lambda t}} \quad \therefore \theta = 120 - 100e^{-\lambda t}$

b) $100 = 120 - 100e^{-0.01t} \quad \Rightarrow -20 = -100e^{-0.01t}$

$\Rightarrow \frac{1}{5} = e^{-0.01t} \quad \Rightarrow \ln\left(\frac{1}{5}\right) = -0.01t \Rightarrow \ln 5 = 0.01t$

$\therefore t = 100\ln 5 \quad \simeq 161 \text{ sec}$

7. A curve is described by the equation

$$x^2 + 4xy + y^2 + 27 = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

A point Q lies on the curve.

The tangent to the curve at Q is parallel to the y-axis.

Given that the x coordinate of Q is negative,

(b) use your answer to part (a) to find the coordinates of Q.

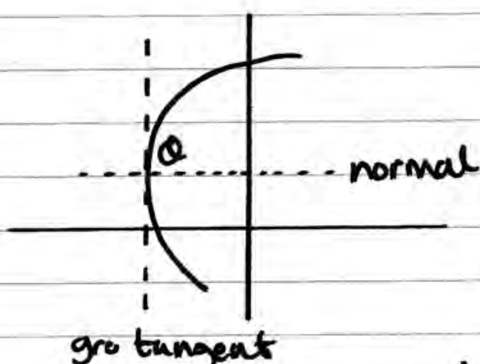
(7)

a) $\frac{dy}{dx} = 2x + 4x\frac{dy}{dx} + 4y + 2y\frac{dy}{dx} = 0$

$(4x+2y)\frac{dy}{dx} = -2x-4y$

$\Rightarrow \frac{dy}{dx} = \frac{-2(x+y)}{2(2x+y)} = -\frac{x+2y}{2x+y}$

b)



$\therefore$ normal gradient $= 0$

gradient tangent $= -\left(\frac{x+2y}{2x+y}\right)$

$\therefore$ gradient normal $= \frac{2x+y}{x+2y}$

$M_n = 0 \Rightarrow 2x+y=0 \Rightarrow y=-2x$

$x^2+4x(-2x)+(-2x)^2+27=0$

$\Rightarrow x^2-8x^2+4x^2+27=0 \quad \Rightarrow -3x^2=-27$

$\therefore x^2=9 \quad \therefore x=-3,3$

$\therefore x=-3,\ y=6 \quad (-3,6)$

8. With respect to a fixed origin O, the line l has equation

$$\mathbf{r} = \begin{pmatrix} 13 \\ 8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \text{ where } \lambda \text{ is a scalar parameter.}$$

The point A lies on l and has coordinates $(3, -2, 6)$.

The point P has position vector $(-p\mathbf{i} + 2p\mathbf{k})$ relative to O, where p is a constant.

Given that vector $\overrightarrow{PA}$ is perpendicular to l,

(a) find the value of p.

(4)

Given also that B is a point on l such that $\angle BPA = 45°$,

(b) find the coordinates of the two possible positions of B.

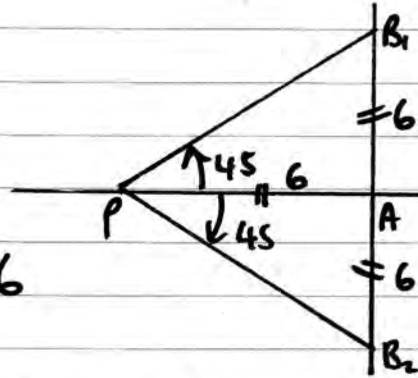
(5)

a) $\overrightarrow{PA} = a - p = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} - \begin{pmatrix} -p \\ 0 \\ 2p \end{pmatrix} = \begin{pmatrix} 3+p \\ -2 \\ 6-2p \end{pmatrix}$

dir $l = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ $\quad$ l perp to $\overrightarrow{PA}$ $= \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3+p \\ -2 \\ 6-2p \end{pmatrix} = 0$

$\Rightarrow 6 + 2p - 4 - 6 + 2p = 0 \Rightarrow 4p = 4 \therefore \underline{p = 1}$

$$B = \begin{pmatrix} 13+2\lambda \\ 8+2\lambda \\ 1-\lambda \end{pmatrix}$$



$$\vec{AP} = \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} \Rightarrow |\vec{AP}| = \sqrt{4^2+2^2+4^2} = 6$$

$$|AB_1| = 6 \quad \left| \begin{pmatrix} 13+2\lambda \\ 8+2\lambda \\ 1-\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} \right| = \left| \begin{matrix} 10+2\lambda \\ 10+2\lambda \\ -5-\lambda \end{matrix} \right|$$

$$\Rightarrow \sqrt{(10+2\lambda)^2+(10+2\lambda)^2+(-5-\lambda)^2} = 6$$

$$\Rightarrow 100+40\lambda+4\lambda^2+100+40\lambda+4\lambda^2+25+10\lambda+\lambda^2 = 36$$

$$9\lambda^2+90\lambda+189=0$$
$$\lambda^2+10\lambda+21=0$$
$$(\lambda+7)(\lambda+3)=0$$
$$\lambda=-7 \quad \lambda=-3$$

$$B_1 = \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix} \quad B_2 = \begin{pmatrix} -1 \\ -6 \\ 8 \end{pmatrix}$$